JOURNAL OF APPROXIMATION THEORY **48**, 201–206 (1986)

The Fourier Projection Is Minimal for Regular Polyhedral Spaces

BRUCE L. CHALMERS

Department of Mathematics, University of California, Riverside, California 92521, USA

Communicated by Oved Shisha

Received November 30, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

Let $V_n = [v_1, ..., v_n]$ be the n-dimensional space of coordinate functions on a set of points $\tilde{v} \subset \mathbb{R}^n$, where $\tilde{v}$ is the set of vertices of a regular convex polyhedron. In this paper it is demonstrated that the Fourier projection F is minimal from $C(\tilde{v})$ onto V_n. As a corollary, F is used to compute the absolute projection constant of any n-dimensional Banach space E_n isometrically isomorphic to $V_n \subset C(\tilde{v})$, examples of which are the well-known cases $E_n = l_n^\infty$, l_n^1.

1. INTRODUCTION

Let $e = e_1, ..., e_n$ be a basis for an arbitrary n-dimensional Banach space E_n with norm $\| \cdot \|$. Then $E_n = ([e], \| \cdot \|) \simeq$ (isometrically isomorphic) $V_n = ([v], \| \cdot \|_\infty)$, where $v = v_1, ..., v_n$ is the n-tuple of coordinate functions for a minimal set of points $\tilde{v}$ in $\mathbb{R}^n$ defined by $\| a \cdot e \| = \| a \cdot v \|_\infty$, $\forall a = a_1, ..., a_n \in \mathbb{R}^n$. Then the norm of an "absolute" minimal projection $P_{\min}(E_n)$ from $C(\tilde{v})$ onto V_n (the norm is computed from the "Lebesgue function" of $P_{\min}$) is the absolute projection constant $\lambda(E_n)$ of E_n.

In this note we will show that when E_n is a regular polyhedral space (e.g., l_1^n and l_∞^n), then we can take $P_{\min}(E_n)$ to be the Fourier projection $F = (1/\| v_1 \|_2^2) \sum_{i=1}^n v_i \otimes v_i$, where $\langle v_i, v_j \rangle_2 = \delta_{ij} \| v_1 \|_2^2$, $1 \leq i, j \leq n$; here $\langle f, g \rangle_2 = \int_{\tilde{v}} fg \, d\mu$, where μ is the unique normalized uniformly distributed measure on $\tilde{v}$ and $F(f) = (1/\| v_1 \|_2^2) \sum_{i=1}^n \langle f, v_i \rangle v_i$. This result unifies and clarifies the results of [4] and [7] where the projection constants are obtained via projections whose connection with the Fourier projections is not at all immediately apparent.

The motivation for examining the Fourier projection in the above context comes from the form of $P_{\min}$ discussed in [1, 2] (note also [3]). There it is demonstrated that $u = (\text{ext} \, | v^{(1)} |) v^{(2)}$ is a fundamental solution for the variational (*)-equation for $P_{\min}$ where $v^{(1)} = vA^{1/2}$ and $v^{(2)} = vA$ for some $n \times n$ matrix A of the form $A = BB'$ where we write $A^{1/2} = B$. (The constants

0021-9045/86 $3.00

c, $\mathbf{c}$, and $\mathbf{c}_p$ in [1] have since been determined in [2] to be zeros; further, any linear combination of fundamental solutions u is also a solution of (*).) But in the case of a regular polyhedral space, the choice of $A = I$ (the identity matrix) yields the maximum number of global maxima for $v^{(1)}$ on which the measure $\text{ext}\,|v^{(1)}|$ can be supported. This observation and the symmetry of $\tilde{v}$ lead us to examine the solution $u = \mu v$. When we do, we find that this solution is the Fourier projection, since we can show that the coordinate functions $\{v_i\}_{i=1}^n$ are biorthogonal, and furthermore we find that it is minimal by comparing its norm to that of [4].

2. Main Results

Let $\tilde{v} = \{\tilde{v}_i\}_{i=1}^k$ be the set of k vertices of a regular polyhedron in $\mathbb{R}^n$, and let $V_n = V_n(\tilde{v}) = [v_1, \ldots, v_n]$ be the associated n-dimensional polyhedral space. $v_j \in \mathbb{R}^k$ is the jth coordinate function on $\tilde{v}$, i.e., $v_{ji} = v_j(\tilde{v}_i) = \tilde{v}_{ij}$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$. The set $\tilde{v}$ is invariant with respect to the group G of automorphisms induced by the regular polyhedron, yielding a subgroup of permutations of $\tilde{v}$.

Remark 1. For $n > 4$ there exist only 3 regular polyhedra, namely the balls of l_n^1 and l_n^∞ (giving rise to the well known spaces l_n^∞ and l_n^1, respectively, for which the projection constants are known [7]) and the regular tetrahedron of the following theorem.

THEOREM 1. *Let $\tilde{v}$ be the set of $n+1$ vertices of the regular tetrahedron T_n in $\mathbb{R}^n$. Then $F = (1/\|v_1\|_2^2) \sum_{i=1}^n v_i \otimes v_i$ is a projection (the Fourier projection) of minimal norm from $C(\tilde{v})$ onto $V_n = [v_1, \ldots, v_n]$ and $\lambda(V_n) = \|F\| = 2n/(n+1)$.*

Proof. We can obtain the coordinate functions $v_1, \ldots, v_n$ and demonstrate that they are biorthogonal by induction as follows. First, for $n = 1$ clearly $v_1^{(1)} = (1, -1)$ and then for $n \geqslant 2$, the coordinate functions ($(n+1)$-tuples) for $\tilde{v}$ are given inductively as follows:

$$\begin{aligned} v_1^{(n)} &= 1 \quad -\frac{1}{n} \quad -\frac{1}{n} \quad \cdots \quad -\frac{1}{n}, \\ v_2^{(n)} &= 0 \quad r_n v_1^{(n-1)}, \\ &\;\;\vdots \\ v_n^{(n)} &= 0 \quad r_n v_{n-1}^{(n-1)}, \end{aligned}$$

where $r_n = \sqrt{n^2 - 1}/n$. Thus, assuming by the induction hypothesis that

$\langle v_i^{(n-1)}, v_j^{(n-1)}\rangle = 0$, $1 \leqslant i \neq j \leqslant n-1$ and that $\sum_{j=1}^{n} v_{ij}^{(n-1)} = 0$, $1 \leqslant i \leqslant n-1$, we see immediately that $\langle v_i^{(n)}, v_k^{(n)}\rangle = 0$, $2 \leqslant i \neq k \leqslant n$, but also $\langle v_1^{(n)}, v_k^{(n)}\rangle = -(r_n/n)\sum_{j=1}^{n} v_{k-1j}^{(n-1)} = 0$ for $2 \leqslant k \leqslant n$. Thus $v_1^{(n)},..., v_n^{(n)}$ are pairwise orthogonal. Further by induction $\langle v_i^{(n)}, v_i^{(n)}\rangle = (n+1)/n$, $1 \leqslant i \leqslant n$. We conclude that F is indeed a projection and we now calculate $\|F\|$ to be $2n/(n+1)$ as follows. By symmetry

$$\begin{aligned}
\|F\| = \sup_{\|f\|_\infty = 1} \|Ff\|_\infty &= \sup_{\|f\|_\infty = 1} (Ff)_1 \\
&= \sup_{\|f\|_\infty = 1} \frac{n}{n+1}\left(\langle f, v_1^{(n)}\rangle v_1^{(n)} + \sum_{i=2}^{n} \langle f, v_i^{(n)}\rangle v_i^{(n)}\right)_1 \\
&= \sup_{\|f\|_\infty = 1} \frac{n}{n+1}\langle f, v_1^{(n)}\rangle v_{11}^{(n)} \\
&= \frac{n}{n+1}\Bigl(1 + \underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{n\text{-times}}\Bigr)1 = \frac{2n}{n+1}.
\end{aligned}$$

Thus F is a minimal projection, since $\lambda(V_n) = \frac{2n}{n+1}$, by [4]. ∎

Theorem 2 below is checked by comparing the easily computed norm of the Fourier projection with the results of [6] and [7].

THEOREM 2. *For $n = 2$, let $\tilde{v}$ be the set of k vertices of the regular polyhedron π_k in $\mathbb{R}^2$. Then $P_{\min}: C(\tilde{v}) \to V_2$ is the associated Fourier projection F and*

$$\lambda(V_2) = \|F\| = \alpha(k) = \begin{cases} \dfrac{4}{k}\cot\dfrac{\pi}{k} & \text{if } \; k = 4l \\[2ex] \dfrac{4}{k}\csc\dfrac{\pi}{k} & \text{if } \; k = 2m \neq 4l; \end{cases}$$

and $\alpha(k) = \alpha(2k)$ if k is odd.

THEOREM 3. *Let $\tilde{v}$ be the set of 2^n vertices of the hypercube (the ball of l_n^∞) in $\mathbb{R}^n$ (whence $l_n^1 \simeq V_n$). Then $F = (1/\|v_1\|_2^2)\sum_{i=1}^{n} v_i \otimes v_i$ is a projection (the Fourier projection) from $C(\tilde{v})$ onto V_n, and $\lambda(V_n) = \lambda(l_n^1) = \|F\| = (2k-1)\,\Gamma(k-\frac{1}{2})/\sqrt{\pi}\Gamma(k)$ if $n = 2k$ or $2k-1$.*

Proof. Let $v_i = v_{i1} v_{i2} \cdots v_{i2^n}$, $1 \leqslant i \leqslant n$ be the coordinate 2^n-tuples (all entries ± 1) and without loss let $v_{i1} = 1$ for all i. Let $\gamma = 1/\|v_1\|_2^2$. Then the $\{v_i\}_{i=1}^n$ are easily seen (e.g. by induction) to be biorthogonal and

$$\begin{aligned}\|F\| = \sup_{\|f\|_\infty = 1} \|Ff\| &= \sup_{\|f\|_\infty = 1} (Ff)_1 \\ &= \sup_{\|f\|_\infty = 1} \gamma \sum_{i=1}^{n} \langle f, v_i \rangle v_{i1} = \sup_{\|f\|_\infty = 1} \gamma \sum_{i=1}^{n} \langle f, v_i \rangle \\ &= \sup_{\|f\|_\infty = 1} \gamma \sum_{i=1}^{n} \sum_{k=1}^{2^n} f_k v_{ik} = \sup_{\|f\|_\infty = 1} \gamma \sum_{k=1}^{2^n} f_k \sum_{i=1}^{n} v_{ik} \\ &= \gamma \sum_{k=1}^{2^n} \left\| \sum_{i=1}^{n} v_{ik} \right\|.\end{aligned}$$

But now there are $2\binom{n}{0}$ ways that $|\sum_{i=1}^n v_{ik}| = n$, $2\binom{n}{1}$ ways that $|\sum_{i=1}^n v_{ik}| = n-2$, $2\binom{n}{2}$ ways that $|\sum_{i=1}^n v_{ik}| = n-4$, etc. Thus since $\gamma = \|v_1\|_2^2 = 2^{-n}$

$$\lambda(l_n^1) = 2\left[\binom{n}{0} n + \binom{n}{1}(n-2) + \binom{n}{2}(n-4) + \cdots + \binom{n}{r} s\right] \Big/ 2^n,$$

where $r = (n-1)/2$ and $s = 1$ if n is odd, and $r = (n/2) - 1$ and $s = 2$ if n is even. Using simple identities, one sees that this answer coincides with that of [7], and thus F is minimal.

THEOREM 4. *Let $\tilde{v}$ be the set of n vertices in the first "octant" of the octahedron (the ball of l_n^1) in $\mathbb{R}^n$ (whence $l_n^\infty \simeq V_n$). Then $F = (1/\|v_1\|_2^2) \sum_{i=1}^n v_i \otimes v_i$ is a projection (the Fourier projection) from $C(\tilde{v})$ onto V_n, and $\lambda(V_n) = \lambda(l_n^\infty) = \|F\| = 1$.*

Proof. It is immediate to check that F is an interpolating projection of norm 1. ∎

Remark 2. If $n = 3$ there are 2 regular convex polyhedra other than l_n^1, l_n^∞, T_n and if $n = 4$ there are 3 such other (see [5]).

We have thus far demonstrated that the Fourier projection is minimal except for these 5 special cases.

THEOREM 5. *Let $\tilde{v}$ be the set of vertices of any one of the 5 special cases of Remark 2. Then $F = (1/\|v_1\|_2^2) \sum_{i=1}^n v_i \otimes v_i$ is a projection (the Fourier projection) from $C(\tilde{v})$ onto V_n. In fact in the case $n = 3$, for the icosahedron $\lambda(V_3) = (1 + \sqrt{5})/2$ and for the dodecahedron $\lambda(V_3) = 0.3(3 + \sqrt{5})$. In the case $n = 4$, for the polyhedron with 24 vertices $\lambda(V_4) = \frac{5}{3}$, for the polyhedron with 120 vertices $\lambda(V_4) = (6\sqrt{5} + 11)/15$, and for the polyhedron with 600 vertices $\lambda(V_4) = (9\sqrt{5} + 22)/25$.*

Proof. Each of these cases is established separately by checking that F is indeed a projection and that $\|F\| = \lambda(V_n)$, where $\lambda(V_n)$ is obtained in

[4]. In the interest of space we will demonstrate this here for only one case, namely that of the the icosahedron. All the other cases can be checked similarly (although the cases of 120 vertices and 600 vertices are quite tedious) and have been checked by the author.

Let v_1, v_2, v_3 be the coordinate functions of the icosahedron, its 12 vertices appropriately positioned so that we have (where $\tau = (\sqrt{5}+1)/2$)

	$\tilde{v}_1$	$\tilde{v}_2$	$\tilde{v}_3$	$\tilde{v}_4$	$\tilde{v}_5$	$\tilde{v}_6$	$\tilde{v}_7$	$\tilde{v}_8$	$\tilde{v}_9$	$\tilde{v}_{10}$	$\tilde{v}_{11}$	$\tilde{v}_{12}$
v_1	0	0	0	0	1	1	-1	-1	τ	τ	$-\tau$	$-\tau$
v_2	τ	τ	$-\tau$	$-\tau$	0	0	0	0	1	-1	1	-1
v_3	1	-1	1	-1	τ	$-\tau$	τ	$-\tau$	0	0	0	0

Then check directly that $\langle v_i, v_j \rangle = 0$, $1 \leqslant i \neq j \leqslant 3$ and that $\langle v_i, v_i \rangle = 4 + 4\tau^2 = \gamma$, $i = 1, 2, 3$. Then by symmetry

$$\begin{aligned}
\|F\| &= \sup_{\|f\|_\infty = 1} \|Ff\| = \sup_{\|f\|_\infty = 1} (Ff)_9 \\
&= \sup_{\|f\|_\infty = 1} \left[\frac{1}{\gamma}(f_5 + f_6 - f_7 - f_8 + \tau f_9 + \tau f_{10} - \tau f_{11} - \tau f_{12})\tau \right. \\
&\qquad \left. + \frac{1}{\gamma}(\tau f_1 + \tau f_2 - \tau f_3 - \tau f_4 + f_9 - f_{10} + f_{11} - f_{12}) 1 \right] \\
&= \sup_{\|f\|_\infty = 1} \frac{1}{\gamma} [\tau(f_1 + f_2 - f_3 - f_4 + f_5 + f_6 - f_7 - f_8) + (\tau^2 + 1)(f_9 - f_{12}) \\
&\qquad + (\tau^2 - 1)(f_{10} - f_{11})] \\
&= \frac{8\tau + 2(\tau^2 + 1) + 2(\tau^2 - 1)}{\gamma} = \frac{\sqrt{5} + 1}{2} = \lambda(V_3)
\end{aligned}$$

from [4]. ∎

Remark 3. Since in [4] there are two different minimal projections for the case of the dodecahedron, we point out that the Fourier projections while minimal are not necessarily unique for regular polyhedral spaces.

References

1. B. L. Chalmers, The (*)-equation and the form of the minimal projection operator, *in* "Approximation Theory IV" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 393–399, Academic Press, New York, 1984.

2. B. L. Chalmers, A variational equation for minimal norm extensions, submitted for publication.
3. B. L. Chalmers, A natural simple projection with norm $\leqslant \sqrt{n}$, *J. Approx. Theory* **32** (1981), 226–232.
4. B. L. Chalmers and B. Shekhtman, Minimal projections and absolute projection constants for regular polyhedral spaces, *Proc. Amer. Math. Soc.*, in press.
5. H. S. Coxeter, "Regular Polytopes," Macmillan, New York, 1963.
6. C. Franchetti and G. Votruba, "Perimeter, Macphail number and projection constant in Minkowski planes," *Boll. Un Mat. Ital. B*(6) **13** (1976), 560–573.
7. B. Grünbaum, Projection constants, *Trans. Amer. Math. Soc.* **95** (1960), 451–465.